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Reflection factors and a two-parameter family of boundary bound states in the sinh–Gordon model

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Abstract. The investigation of boundary breather states of the sinh–Gordon model restricted to a half-line is revisited. Properties of the classical boundary breathers for the two-parameter family of integrable boundary conditions are reviewed and extended. The energy spectrum of the quantized boundary states is computed, firstly by using a bootstrap technique and, subsequently using a WKB approximation. Requiring that the two descriptions of the spectrum agree with one another allows a determination of the relationship between the boundary parameters, the bulk coupling constant, and the two parameters appearing in the reflection factor describing the scattering of the sinh–Gordon particle from the boundary. These calculations had been performed previously for the case in which the boundary conditions preserve the bulk Z_2 symmetry of the model. The significantly more difficult case of general boundary conditions which violate the bulk symmetry is treated in this paper. The results clarify the weak–strong-coupling duality of the sinh–Gordon model with integrable boundary conditions.

1. Introduction

In the study of the sinh–Gordon model restricted to a half-line by integrable boundary conditions, an important issue is to determine how the reflection factor depends upon the two parameters introduced at the boundary. Some progress was recently made in tackling this question [3, 6]. Namely, the idea of [6] was to compute the bound-state spectrum of the model in two different ways, firstly by using a boundary bootstrap principle, and secondly by quantizing the classical boundary breather states using a WKB approach [8, 17]. Each method provides an independent description of the energy spectrum of the boundary states, and comparing the two provides information relating the boundary parameters and the bulk coupling to the reflection factor. Clearly, one expects the most general situation where the two boundary parameters are independent to be technically more involved. It is therefore natural—as was done in [6]—to first implement the programme outlined above in the special case where the bulk symmetry $\phi \to -\phi$ is preserved at the boundary, by requiring the two parameters to be equal. A further step was taken in [3], where a perturbative calculation to lowest order in the bulk coupling and to first order in the difference of the two boundary parameters was used to make an informed guess as to the general dependence of the reflection factor on those boundary parameters. The results of the present paper, which extends the analysis in [6] to the case where the two boundary parameters are different, will underpin that guess. Finally, a version of a weak–strong-coupling duality transformation appropriate to the model with

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boundary conditions was also proposed in [3] and the conclusions drawn from our paper give further support to that conjecture.

Following a brief introduction to the sinh–Gordon model with integrable boundary conditions in section 1, we recall the classical breather solutions in section 2 and describe the properties and facts we need for the semiclassical quantization. In section 3, we review the boundary state bootstrap and use it to derive a formula for the boundary state energy spectrum. The adapted Dashen–Hasslacher–Neveu method is described in section 4 and used to provide an alternative calculation of the boundary state spectrum in terms of the boundary parameters. Our results and some additional remarks are summarized in section 5.

2. The sinh–Gordon model on the half-line

The sinh–Gordon model describes a single real scalar field ϕ in 1 + 1 dimensions with exponential self-interaction. The field equation is

$$\partial_t^2\phi - \partial_x^2\phi + \frac{\sqrt{8}m^2}{\beta}\sinh(\sqrt{2}\beta\phi) = 0 \tag{2.1}$$

where m and β are parameters and we have used normalizations customary in affine Toda field theories of which the sinh–Gordon model is the simplest [2]. The dimensional mass parameter m will be set to unity for convenience.

In contrast to the sine–Gordon model, with its soliton and breather solutions, the sinh–Gordon model is at first sight relatively uninteresting. There is a constant vacuum solution $\phi = 0$ and, in the quantum theory, the small oscillations around this vacuum correspond to the sinh–Gordon particle. In the bulk, the spectrum of the model consists of a single species of scalar particle interacting with itself. Nevertheless, however simple the model may appear, it is not easy to analyse it directly [18], and much of what is known about it has been deduced from features of the lightest breather in the sine–Gordon theory.

The sinh–Gordon model is integrable which implies, in particular, that there are infinitely many mutually commuting, independent conserved charges $Q_{\pm s}$, where s is any odd integer, and the S-matrix describing the scattering of two sinh–Gordon particles with relative rapidity Θ is conjectured to be given by [10, 19],

$$S(\Theta) = -\frac{1}{(B)_\Theta(2-B)_\Theta}. \tag{2.2}$$

In (2.2) we have used the convenient block notation [2]

$$(x)_\Theta = \frac{\sinh\,(\Theta/2 + \mathrm{i}\pi x/4)}{\sinh\,(\Theta/2 - \mathrm{i}\pi x/4)} \tag{2.3}$$

and the coupling constant B is related to the bare coupling constant β by $B = 2\beta^2/(4\pi + \beta^2)$. For compactness, we will generally omit the subscript Θ from the block notation since it is generally clear from the context what is intended.

The sinh–Gordon model can be restricted to the left half-line $-\infty \leqslant x \leqslant 0$ without losing integrability by imposing the boundary condition

$$\partial_x\phi|_0 = \frac{\sqrt{2}m}{\beta}\left(\varepsilon_0 \mathrm{e}^{-(\beta/\sqrt{2})\phi(0,t)} - \varepsilon_1 \mathrm{e}^{(\beta/\sqrt{2})\phi(0,t)}\right) \tag{2.4}$$

where ε_0 and ε_1 are two additional parameters [12, 15]. Again, m sets the scale but is taken to be unity for convenience in what follows. This set of boundary conditions generally breaks the reflection symmetry $\phi \to -\phi$ of the model, although the symmetry is preserved when $\varepsilon_0 = \varepsilon_1 \equiv \varepsilon$.

Assuming factorization, the description of the sinh–Gordon particles on the half-line requires not only the two-particle scattering amplitude (2.2), but also the amplitude for the reflection of a single particle from the boundary. This reflection amplitude was deduced from the lowest breather reflection amplitude in the sine–Gordon model by analytic continuation in the coupling constant (i.e. the continuation $\lambda \to -2/B$ in the notation of [12]). Using the breather reflection amplitudes calculated by Ghoshal [13], the analytic continuation leads to†

$$K_q(\theta, \varepsilon_0, \varepsilon_1, \beta) = \frac{(1)(2 - B/2)(1 + B/2)}{(1 - E(\varepsilon_0, \varepsilon_1, \beta))(1 + E(\varepsilon_0, \varepsilon_1, \beta))(1 - F(\varepsilon_0, \varepsilon_1, \beta))(1 + F(\varepsilon_0, \varepsilon_1, \beta))} \tag{2.5}$$

where we are again using the block notation from (2.3) but, here, θ represents the rapidity of a single particle. When the bulk reflection symmetry is preserved one of the two parameters E or F vanishes. Without loss of generality, we shall take the vanishing parameter in the symmetric case to be F. All reflection factors satisfy the crossing-unitarity relation which, in the case of scalar reflection factors, reads,

$$K_q\left(\theta + \tfrac{1}{2}\mathrm{i}\pi\right) K_q\left(\theta - \tfrac{1}{2}\mathrm{i}\pi\right) S(2\theta) = 1. \tag{2.6}$$

Actually, equation (2.5) is the simplest solution to the crossing-unitarity relation using the S-matrix (2.2), taking into account the independently calculated classical limit of the reflection factors. In [5] the classical reflection factor was found to be given by the formula

$$K_c(\theta, \varepsilon_0, \varepsilon_1, \beta) = -\frac{(1)^2}{(1 - a_0 - a_1)(1 + a_0 + a_1)(1 - a_0 + a_1)(1 + a_0 - a_1)} \tag{2.7}$$

in which it was convenient to use an alternative expression for the boundary parameters, namely

$$\epsilon_0 = \cos \pi a_0 \qquad \epsilon_1 = \cos \pi a_1. \tag{2.8}$$

The formula is well defined provided we select the ranges $0 \leqslant a_i \leqslant 1$ for $i = 0, 1$, to ensure a one-to-one correspondence between the alternative parameters. Clearly, equation (2.5) has the correct limit provided

$$E \to a_0 + a_1 \qquad F \to a_0 - a_1. \tag{2.9}$$

Recently, on the basis of a perturbative calculation, the relationship between the various parameters was conjectured to be [3],

$$E = (a_0 + a_1)(1 - B/2) \qquad F = (a_0 - a_1)(1 - B/2). \tag{2.10}$$

The principal purpose of this paper is to provide further evidence for these formulae.

It was noted in [6] that contrary to the situation on the whole line, the sinh–Gordon equation restricted to a half-line by integrable boundary conditions has non-singular, finite-energy, breather solutions. These solutions were described in some detail in [6], particularly for the special case in which $\epsilon_0 = \epsilon_1$. Here, we shall concentrate on the solutions containing two independent boundary parameters. The existence of these special solutions and their associated states makes it clear that the sinh–Gordon model has a rather more interesting structure than one would be led to believe on the basis of bulk calculations.

† In Ghoshal's notation $E = B\eta/\pi$, $F = \mathrm{i}B\vartheta/\pi$.

3. Boundary breathers

The boundary breathers may be conveniently described following Hirota's prescription [14]. Generally, provided we set

$$\phi = -\frac{\sqrt{2}}{\beta}\ln\frac{\tau_+}{\tau_-} \tag{3.1}$$

and choose the two τ-functions as follows:

$$\tau_\pm = 1 \pm (E_1 + E_2 + E_3) + (A_{12}E_1E_2 + A_{13}E_1E_3 + A_{13}E_1E_3) \pm A_{12}A_{13}A_{23}E_1E_2E_3 \tag{3.2}$$

with

$$\begin{aligned} E_p &= \mathrm{e}^{a_p x + b_p t + c_p} \qquad & a_p &= 2\cosh\rho_p \\ b_p &= 2\sinh\rho_p \qquad & A_{pq} &= \tanh^2\left(\tfrac{1}{2}(\rho_p - \rho_q)\right) \end{aligned} \tag{3.3}$$

then we have just enough freedom to accommodate the general boundary conditions (2.4). In effect, with general boundary conditions, the solutions have the flavour of a 'breather' superimposed on a stationary 'soliton', borrowing the language of the sine–Gordon model. In detail, it is enough to choose E_1 and E_2 to be complex conjugate partners and periodic in t, and to take E_3 to be real and independent of t; thus, setting $\rho_1 = -\rho_2 = \mathrm{i}\rho$ and $\rho_3 = 0$, one has:

$$\begin{aligned} E_1 &= \mathrm{e}^{2x\cos\rho + 2\mathrm{i}t\sin\rho + c} = E_2^* \qquad & E_3 &= \mathrm{e}^{2x+d} \\ A_{12} &= -\tan^2\rho & A_{13} &= A_{23} = -\tan^2(\rho/2). \end{aligned} \tag{3.4}$$

The period of the breather is $\pi/\sin\rho$. To match the boundary conditions, c and d need to be determined. It is convenient (actually just a change of origin in t) to take c to be real. Then, the boundary conditions (2.4) are satisfied provided

$$\mathrm{e}^c = \frac{s}{\tan\rho} \qquad \mathrm{e}^d = \frac{r}{\tan^2(\rho/2)} \tag{3.5}$$

where

$$\begin{aligned} r &= \frac{\sin(\pi a_0/2) - \sin(\pi a_1/2)}{\sin(\pi a_0/2) + \sin(\pi a_1/2)} \\ s^2 &= \frac{1+\cos\rho}{1-\cos\rho}\,\frac{\cos[\pi(a_0+a_1)/2] + \cos\rho}{\cos[\pi(a_0+a_1)/2] - \cos\rho}\,\frac{\cos[\pi(a_0-a_1)/2] - \cos\rho}{\cos[\pi(a_0-a_1)/2] + \cos\rho}. \end{aligned} \tag{3.6}$$

In (3.6) we have made use of the alternative parametrization (2.8). There are other ways of writing (3.6) which are useful when it comes to evaluating certain integrals. For example, setting $q = \tan(\rho/2)$ and $q_\pm = \tan(\pi a_\pm/2)$, where $a_\pm = (a_0 \pm a_1)/2$, we have

$$s^2 = \frac{1}{q^2}\,\frac{1 - q^2q_+^2}{1 - q^{-2}q_+^2}\,\frac{1 - q^{-2}q_-^2}{1 - q^2q_-^2} \qquad r = \frac{q_-}{q_+}. \tag{3.7}$$

For symmetrical boundary conditions with $\varepsilon_0 = \varepsilon_1 = \varepsilon$, the terms containing E_3 are not required (effectively $d \to -\infty$), and the other pieces of the τ-functions collapse to

$$\tau_\pm = 1 \pm 2\cos(2t\sin\rho)\,\mathrm{e}^{2x\cos\rho}\,\frac{1}{\tan\rho}\sqrt{\frac{\varepsilon+\cos\rho}{\varepsilon-\cos\rho}} - \mathrm{e}^{4x\cos\rho}\left(\frac{\varepsilon+\cos\rho}{\varepsilon-\cos\rho}\right). \tag{3.8}$$

The expression (3.8) is quite easy to analyse. In particular, the solutions ought to be real and have no singularities in the region $x < 0$. It was pointed out in [6] that these requirements are met provided the parameters ε and ρ are suitably restricted:

$$-1 < \varepsilon < 0 \qquad \text{and} \qquad \cos\rho < -\varepsilon. \tag{3.9}$$

It is interesting that the upper limit $\cos\rho = -\varepsilon$ corresponds to a choice of frequency at which the amplitude of the solution has collapsed to zero yielding the vacuum configuration, $\phi = 0$. This is quite unlike the bulk breathers of the sine–Gordon theory which approach the vacuum as their frequencies approach zero. On the other hand, this behaviour of the boundary breathers is closer to that of a standard harmonic oscillator of a given frequency whose classical amplitude may be arbitrarily small. Once quantized, the energy spectrum of the oscillator depends only on its frequency and the zero-point energy is there to remind us that an oscillator of arbitrarily small amplitude is distinct from the vacuum. Similarly, we expect that the sinh–Gordon boundary breathers will have a non-zero ground state energy.

For general boundary conditions the properties of the solutions are less amenable to analysis. However, numerical investigation of the general boundary breathers indicates that they are non-singular in the region $x < 0$ provided

$$\cos\tfrac{1}{2}\pi(a_0+a_1) < 0 \qquad 0 < \cos\rho < -\cos\tfrac{1}{2}\pi(a_0+a_1) \qquad \cos\tfrac{1}{2}\pi(a_0-a_1) > 0. \tag{3.10}$$

Again, the breathers have frequencies bounded below because the parameters are restricted. We shall take the parameters to satisfy $a_0 \geqslant a_1$ (since one must be larger than the other unless they are equal, we take the larger to be a_0), and to lie within the region defined by

$$0 \leqslant a_0 \qquad a_1 \leqslant 1 \qquad 0 \leqslant a_- \leqslant \tfrac{1}{2} \qquad \tfrac{1}{2} \leqslant a_+ \leqslant 1 \qquad \pi(1-a_+) \leqslant \rho \leqslant \tfrac{1}{2}\pi. \tag{3.11}$$

As we have noted, the boundary breathers for boundary conditions preserving the symmetry of the sinh–Gordon equation are included as the special case $a_0 = a_1$.

The energy functional of the sinh–Gordon model incorporating the boundary condition (2.4) is given by

$$\mathcal{E}[\phi] = \int_{-\infty}^{0} \mathrm{d}x \left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\phi'^2 + \frac{2}{\beta^2}\left(\cosh(\sqrt{2}\beta\phi) - 1\right)\right) + \frac{2}{\beta^2}\left(\varepsilon_0\left(\mathrm{e}^{-(\beta/\sqrt{2})\phi(0,t)} - 1\right) + \varepsilon_1\left(\mathrm{e}^{(\beta/\sqrt{2})\phi(0,t)} - 1\right)\right) \tag{3.12}$$

but it is most easily calculated in terms of the τ functions as a boundary term [7],

$$\mathcal{E}[\phi] = \frac{2}{\beta^2}\left(\varepsilon_0\left(\frac{\tau_+}{\tau_-} - 1\right) + \varepsilon_1\left(\frac{\tau_-}{\tau_+} - 1\right) - \left(\frac{\tau_+'}{\tau_+} + \frac{\tau_-'}{\tau_-}\right)\right)\Bigg|_{x=0}. \tag{3.13}$$

Using this, the energies of the boundary breathers were calculated in [6] and are

$$\mathcal{E} = \frac{4}{\beta^2}\left(-2 - 2\cos\rho + \left(\sin\tfrac{1}{2}\pi a_0 + \sin\tfrac{1}{2}\pi a_1\right)^2\right). \tag{3.14}$$

For the symmetric case with $a_0 = a_1$ the energy simplifies to

$$\mathcal{E}_{\text{breather}} = \frac{8}{\beta^2}(-\cos\rho - \varepsilon). \tag{3.15}$$

In the quantum field theory, the continuum of boundary breather solutions is expected to lead to a discrete spectrum of boundary bound states. To estimate this spectrum we shall follow the method described in [6] which adapts the techniques developed by Dashen, Hasslacher and Neveu (DHN) [8]. Although it is not obvious that this method gives an exact result, it is clear that the result is non-perturbative, in the sense of being an all-orders computation in perturbation theory in terms of the bulk coupling constant and the boundary parameters.

One of the ingredients to the DHN prescription is the classical action computed over a single period of the boundary breather. This quantity is relatively straightforward to calculate for the symmetric boundary condition but, as far as we can see, is not tractable analytically for the general breather. Nevertheless, we have conjectured the result and checked it numerically in two different ways. The principal difficulty lies in calculating the integral representing the kinetic energy over a single period. We maintain this should be given by a simple expression linear in a_+, that is

$$I = \int_0^T \mathrm{d}t \int_{-\infty}^0 \mathrm{d}x\, \dot{\phi}^2 = \frac{8\pi}{\beta^2}(\rho - \pi(1 - a_+)) \tag{3.16}$$

where $T = \pi/\sin\rho$ is the period of the general breather. When $a_0 = a_1$, equation (3.16) agrees with the result given in [6]; using Maple it is possible to check that the integral is independent of a_- by differentiating the integrand with respect to a_- and calculating the resulting integral numerically, obtaining 0; it is possible to check directly, again by numerical integration within the ranges of parameters (3.10), that I depends linearly on both ρ and a_+. Some details of these computations will be given in the appendix. The conjectured expression (3.16) seems to us to be an astonishing result for which we are quite unable to find an analytical derivation, despite its beguiling simplicity.

4. The boundary bootstrap

For certain ranges of the parameters E and F the particle reflection amplitude (2.5) has simple poles at particular imaginary values of θ on the physical strip, $0 < \mathrm{Im}(\theta) < \pi/2$. These are expected to be due to the propagation of virtual excited boundary states, although there are also other potential explanations via generalized Coleman–Thun mechanisms [9, 16]. The reason for this is the following. Once the boundary condition breaks the bulk symmetry of the model, the lowest-energy field configuration is no longer $\phi = 0$. This means that in the bulk there will be effective odd-point couplings in addition to the standard even-point couplings [3], and these may be used to construct loop Feynman diagrams 'attached' to the boundary, some of which may generate poles in the reflection factor K_q. This possibility is difficult to analyse and we are unable to pursue it here. We shall simply assume that the poles are due to virtual excited boundary bound states without concerning ourselves with their dynamical origin. The amplitudes for the reflection of the sinh–Gordon particle from these excited boundary states are obtained by the boundary bootstrap [4, 11, 12]. When the reflection factor (2.5) has a pole at $\theta = \mathrm{i}\psi$ with $0 < \psi < \pi/2$ then the reflection factor corresponding to the associated excited boundary state is calculated via the relation

$$K_1(\theta) = K_0(\theta)S(\theta - \mathrm{i}\psi)S(\theta + \mathrm{i}\psi) \tag{4.1}$$

where $S(\theta)$ is the two-particle S-matrix (2.2) and $K_0(\theta)$ is the ground state reflection factor. Also, since energy is conserved, the energy of the excited boundary state relative to the ground state is given by

$$\mathcal{E}_1 = \mathcal{E}_0 + m(\beta)\cos\psi \tag{4.2}$$

where $m(\beta)$ is the mass of the sinh–Gordon particle.

As a parenthetical remark it is worth mentioning the consequences of this bootstrap procedure for a free, real scalar field of mass m with a linear boundary condition at $x = 0$:

$$\partial_x\phi|_0 = -m\lambda\phi.$$

Its S-matrix is unity, but the reflection factor is given by

$$K = \frac{\mathrm{i}\sinh\theta + \lambda}{\mathrm{i}\sinh\theta - \lambda} = -\frac{1}{(1+2a)(1-2a)}$$

where $\lambda = \cos a\pi$ in the second formula. This has a pole which indicates (provided $-m < \lambda < 0$ or $1 > a > \frac{1}{2}$) a boundary breather of a fixed frequency ω, given by $\omega^2 = m^2 - \lambda^2 = m^2 \sin^2 a\pi$. Explicitly, the appropriate normalizable solution to the Klein–Gordon equation and the boundary condition is $\phi = A\mathrm{e}^{-\lambda x}\cos\omega t$. Repeated application of the bootstrap equations leads to a tower of boundary states, each with the same reflection factor, but with energies given by

$$\mathcal{E}_n = \mathcal{E}_0 + n\omega.$$

This is just as one would expect for a harmonic oscillator attached to the boundary although an alternative dynamical argument would be needed to determine the ground state energy $\mathcal{E}_0 = \omega/2$.

Returning to the sinh–Gordon case, let us begin by considering the classical reflection factor (2.7) in the light of the boundary breather parameter restrictions (3.10). It is immediately clear that because of the restrictions on the parameters only one of the four factors in the denominator of (2.7), namely $(1 - a_0 - a_1)$ has a pole in the physical strip (remember, we have taken the principal values $1 > a_+ > \frac{1}{2}, \frac{1}{2} > a_- > 0$). We may also recall the perturbative result reported in [3] which implies that at least for sufficiently small coupling E and F do not roam far from their classical values. Bearing in mind the classical limits (2.9), this remark suggests that we may consider the poles associated with E alone, ignoring F.

The remainder of the discussion in this section follows closely the calculations reported in [6] but, for the sake of completeness and small changes of notation, they will be repeated here. The regions in E where the amplitude (2.5) has poles on the physical strip are

$$\text{I:}\quad 2 > E > 1 \qquad \text{and} \qquad \text{II:}\quad -2 < E < -1 \tag{4.3}$$

since $0 \leqslant B \leqslant 2$, the other factors never have poles in the physical strip. In region I, $\psi = \pi(E-1)/2$ and, using (4.1), we derive the reflection factor for the first excited state,

$$K_1 = \frac{K_S}{(1-E)(1+E)} \frac{(1+E+B)(1-E-B)}{(1-E+B)(1+E-B)} \tag{4.4}$$

where the 'spectator' factors have been lumped together in

$$K_S = \frac{(1)(1+B/2)(2-B/2)}{(1-F)(1+F)}. \tag{4.5}$$

The reflection factor (4.4), in turn, has a new pole at $\psi = \pi(E-1-B)/2$, provided $B < E-1$, indicating another excited state whose reflection factor is

$$K_2 = \frac{K_S}{(1-E+B)(1+E-B)} \frac{(1+E+B)(1-E-B)}{(1-E+2B)(1+E-2B)}. \tag{4.6}$$

Continuing the procedure leads to a set of excited states with associated reflection factors given by

$$K_n = \frac{K_S}{(1-E+(n-1)B)(1+E-(n-1)B)} \frac{(1+E+B)(1-E-B)}{(1-E+nB)(1+E-nB)}. \tag{4.7}$$

Note that the pole corresponding to the $(n+1)$th state will be within the correct range provided E satisfies $2 > E > 1+nB$. Thus, for a given E and B there can be at most a finite number of bound states, and possibly none. Note too that the reflection factor for scattering from the nth bound state also contains a pole corresponding to the $(n-1)$th bound state. Since the factor K_S is merely a spectator, at no stage will poles with F-dependent positions enter the game if they did not do so at the start.

The energies $\mathcal{E}_n$ of the boundary states are found by repeatedly applying (4.2). They are given by

$$\mathcal{E}_{n+1} = \mathcal{E}_n + m(\beta)\cos\tfrac{1}{2}\pi(nB - E + 1). \tag{4.8}$$

This is the result that we will compare with the quantization of the classical breather spectrum in order to determine how E, F and $m(\beta)$ depend upon a_0 and a_1.

The poles in region II do not represent a new set of states. In the discussion section we shall make some comments concerning the relative roles of E and F.

5. Semiclassical quantization

The first step in carrying out the semiclassical calculation is to solve the sinh–Gordon equation linearized in the presence of the boundary breathers. Setting $\phi = \phi_0 + \eta$, where ϕ_o is a classical breather, the linear wave equations which ought to be satisfied by the fluctuations are

$$\frac{\partial^2\eta}{\partial t^2} - \frac{\partial^2\eta}{\partial x^2} + 4\eta\cosh\sqrt{2}\beta\phi_0 = 0 \qquad \left(\frac{\partial\eta}{\partial x} + \eta\left[\varepsilon_0 e^{-\beta\phi_0/\sqrt{2}} + \varepsilon_1 e^{\beta\phi_0/\sqrt{2}}\right]\right)_{x=0} = 0. \tag{5.1}$$

It is convenient to solve (5.1) by perturbing (3.1). In other words, we may take

$$\eta = \frac{\tau_-\delta\tau_+ - \tau_+\delta\tau_-}{\tau_+\tau_-} \tag{5.2}$$

with $\delta\tau_\pm$ chosen, in turn, by adding a pair of 'small' exponential terms to (3.2) as follows. Take e_1 and e_2 defined by

$$e_1 = \lambda_1\, e^{-i\omega t + ikx} \qquad e_2 = \lambda_2\, e^{-i\omega t - ikx} \qquad \omega^2 - k^2 = 4$$

where λ_1 and λ_2 are infinitesimally small. We use Hirota's method again, but with five basic ingredients this time, instead of the previous three, keeping only the terms linear in e_1 and e_2. Thus, we write,

$$\delta\tau_\pm = \sum_{p=1,2} e_p\big(\pm 1 + (\mu_{p1}E_1 + \mu_{p2}E_2 + \mu_{p3}E_3) \pm (\mu_{p1}\mu_{p2}A_{12}E_1E_2 + \mu_{p1}\mu_{p3}A_{13}E_1E_3 + \mu_{p2}\mu_{p3}A_{23}E_2E_3) + \mu_{p1}\mu_{p2}\mu_{p3}A_{12}A_{13}A_{23}E_1E_2E_3\big). \tag{5.3}$$

The new coefficients μ_{pq} are obtained from the general formulae (3.3) and given by

$$\begin{aligned} \mu_{11} &= \frac{1}{\mu_{22}} = \frac{ik\cos\rho - \omega\sin\rho - 2}{ik\cos\rho - \omega\sin\rho + 2} \\ \mu_{12} &= \frac{1}{\mu_{21}} = \frac{ik\cos\rho + \omega\sin\rho - 2}{ik\cos\rho + \omega\sin\rho + 2} \\ \mu_{13} &= \frac{1}{\mu_{23}} = \frac{ik - 2}{ik + 2}. \end{aligned} \tag{5.4}$$

Matching the boundary condition at $x = 0$ fixes the ratio λ_2/λ_1 to be†

$$K_B = \frac{\lambda_2}{\lambda_1} = \left(\frac{ik - 2\cos\rho}{ik + 2\cos\rho}\right)^2 \frac{ik-2}{ik+2}\,\frac{ik - 2c_+}{ik + 2c_+}\,\frac{ik + 2c_-}{ik - 2c_-} \tag{5.5}$$

† Remark. The right-hand side of the corresponding result in [6], equation (5.5), has a misprint and the printed formula ought to be inverted to obtain the correct result for K_B.

where $c_\pm = \cos\pi(a_0 \pm a_1)/2$. The result (5.5) is not intended to be obvious; its derivation relied heavily on using a symbolic algebraic manipulation language—in our case, Maple. In the limit $x \to -\infty$, the fluctuation η is a superposition of left- and right-moving plane waves,

$$\eta \sim \lambda_1\, \mathrm{e}^{-\mathrm{i}\omega t}\left(\mathrm{e}^{\mathrm{i}kx} + K_B\, \mathrm{e}^{-\mathrm{i}kx}\right) \tag{5.6}$$

and the relative phase of these waves defines the classical reflection factor corresponding to the boundary breather. Taking $\cos\rho = -c_+$, the breather collapses to the static ground state solution and the reflection factor collapses to

$$K_0 = \frac{\mathrm{i}k-2}{\mathrm{i}k+2}\,\frac{\mathrm{i}k+2c_+}{\mathrm{i}k-2c_+}\,\frac{\mathrm{i}k+2c_-}{\mathrm{i}k-2c_-} \tag{5.7}$$

which, with $k = 2\sinh\theta$, is simply an alternative form (i.e. without using the 'block' notation) of the expression (2.7).

The period $T = \pi/\sin\rho$ of the boundary breather defines the 'stability angles' via

$$\eta(t+T, x) = \mathrm{e}^{-\mathrm{i}\nu}\eta(t,x) \equiv \mathrm{e}^{-\mathrm{i}\omega T}\eta(t,x) \tag{5.8}$$

and the field-theoretical version of the WKB approximation makes use of the stability angles together with a regulator to calculate a quantum action. If there are no boundaries, it is natural to add some artificially to render the spectrum discrete and facilitate the necessary calculations. One way, the simplest and most commonly used, would be to place the field theory in an interval $[-L, L]$ with periodic boundary conditions and to manipulate the sum over the discrete stability angles. However, since we have one boundary already prescribed, we need to do something else as suggested in [6]. It is convenient to treat the sinh–Gordon model in the interval $[-L, 0]$ and to impose the Dirichlet condition $\eta(t, -L) = 0$. Since the limit $L \to \infty$ will be taken eventually, the stability angles for the boundary breather (ν_B), or the vacuum solution (ν_0) are effectively determined by the reflection factors given in (5.6) or (5.7), respectively. A potentially more interesting calculation but one which we do not yet have the machinery to carry out, would be to consider two sets of two-parameter boundary conditions. However, the breathers we are considering are inadequate for that.

Following [8, 17] we need to calculate a sum over the stability angles and use it to correct the classical action. Thus,

$$\Delta = \tfrac{1}{2}\sum(\nu_B - \nu_0) \equiv \tfrac{1}{2}T\sum\left(\sqrt{k_B^2+4} - \sqrt{k_0^2+4}\right) \tag{5.9}$$

where k_B and k_0 are the sets of (discrete) solutions to

$$\begin{aligned}
\mathrm{e}^{2\mathrm{i}k_B L} &= -\left(\frac{\mathrm{i}k_B + 2\cos\rho}{\mathrm{i}k_B - 2\cos\rho}\right)^2 \frac{\mathrm{i}k_B+2}{\mathrm{i}k_B-2}\,\frac{\mathrm{i}k_B+2c_+}{\mathrm{i}k_B-2c_+}\,\frac{\mathrm{i}k_B-2c_-}{\mathrm{i}k_B+2c_-}\\
\mathrm{e}^{2\mathrm{i}k_0 L} &= -\frac{\mathrm{i}k_0+2}{\mathrm{i}k_0-2}\,\frac{\mathrm{i}k_0-2c_+}{\mathrm{i}k_0+2c_+}\,\frac{\mathrm{i}k_0-2c_-}{\mathrm{i}k_0+2c_-}.
\end{aligned} \tag{5.10}$$

Once Δ is known, the quantum action is defined by

$$S_{\mathrm{qu}} = S_{\mathrm{cl}} - \Delta \tag{5.11}$$

where the classical action is readily calculated from the kinetic energy integral (3.16) and the total energy (3.14):

$$S_{\mathrm{cl}} = \int_0^T \mathrm{d}t \int_{-\infty}^0 \mathrm{d}x\, \mathcal{L} = \frac{8\pi}{\beta^2}\left(\rho - \pi(1-a_+) + \frac{1}{\sin\rho}\left(\cos\rho + 1 - \tfrac{1}{2}(1-c_+)(1+c_-)\right)\right). \tag{5.12}$$

We shall proceed along the lines described in [6] noting that for large k the solutions to either of (5.10) are close to

$$k_n = \left(n + \tfrac{1}{2}\right)\frac{\pi}{L}$$

and so it is reasonable to set $(k_B)_n = (k_0)_n + \kappa((k_0)_n)/L$ where, for L large, the function κ is given approximately by

$$\mathrm{e}^{2\mathrm{i}\kappa(k)} = \left(\frac{\mathrm{i}k + 2\cos\rho}{\mathrm{i}k - 2\cos\rho}\,\frac{\mathrm{i}k + 2c_+}{\mathrm{i}k - 2c_+}\right)^2. \tag{5.13}$$

Interestingly, all dependence upon c_- has dropped out and therefore, from this point on, the calculation is identical to the corresponding part of the calculation presented in [6] with $\varepsilon = \cos\pi a$ replaced by c_+.

In terms of κ the expression (5.9) is rewritten

$$\Delta \sim \frac{T}{2L}\sum_{n\geqslant 0}\frac{(k_0)_n\kappa((k_0)_n)}{\sqrt{(k_0)_n^2 + 4}} + \mathrm{O}(1/L^2)$$

and this, in turn, as $L \to \infty$ can be converted to a convenient (but actually divergent) integral,

$$\Delta = \frac{T}{2\pi}\int_0^\infty \mathrm{d}k\,\frac{k\kappa(k)}{\sqrt{k^2+4}} \tag{5.14}$$

with which we shall have to deal. Note that κ vanishes when $\cos\rho = -c_+$. Integrating (5.14) by parts we find

$$\Delta = \frac{T}{2\pi}\left(\kappa\sqrt{k^2+4}\,\Big|_0^\infty - \int_0^\infty \mathrm{d}k\,\frac{\mathrm{d}\kappa}{\mathrm{d}k}\sqrt{k^2+4}\right) \tag{5.15}$$

where

$$\frac{\mathrm{d}\kappa}{\mathrm{d}k} = \frac{4\cos\rho}{k^2 + 4\cos^2\rho} + \frac{4c_+}{k^2 + 4c_+^2} \tag{5.16}$$

and we note that with a suitable choice of branch

$$\kappa \sim -\frac{4\cos\rho}{k} - \frac{4c_+}{k} \qquad \text{as} \quad k \to \infty. \tag{5.17}$$

Using (5.17) and recalling that $\cos\rho < -c_+$, we deduce that κ approaches zero from above as $k \to \infty$. Also, from (5.16) it is clear that the derivative of κ is positive near $k = 0$ but negative as $k \to \infty$. Hence, the first term in (5.15) is well defined and the appropriate branch of κ has $\kappa(0) = 0$. On the other hand, the derivative of κ is not decaying sufficiently rapidly to ensure that the second term in (5.15) is finite. However, this was to be expected since a perturbative analysis of the sinh–Gordon model confined to a half-line needs mass and boundary counter terms to remove logarithmic divergences (which would be removed automatically by normal-ordering the products of fields in the bulk theory). With this in mind, the integral remaining in (5.15) should be replaced by

$$\int_0^\infty \mathrm{d}k\,\sqrt{k^2+4}\left(\frac{4\cos\rho}{k^2+4\cos^2\rho} - \frac{4\cos\rho}{k^2+4} + \frac{4c_+}{k^2+4c_+^2} - \frac{4c_+}{k^2+4}\right) \tag{5.18}$$

the first counter-term removing the bulk divergence and the second being there to remove a similar divergence associated with the boundary. In effect, we are regarding the parameter a as describing the bare coupling which appears in the boundary part of the Lagrangian once it is written in terms of normal-ordered products of fields. The counter-terms clearly respect the symmetry and the whole expression vanishes when $\rho = \pi(1-a_+)$. The integrals in (5.18) need to be treated carefully noting that $\cos\rho > 0$ but $c_+ < 0$.

Besides the towers of real solutions to (5.10), there is also a discrete set of solutions for which k_0 and k_B are pure imaginary. These were not discussed in [6], but including them in the argument would not have altered the conclusions, as we shall show. First, note that as $L \to \infty$ the imaginary solutions which survive are either zeros or poles of the right-hand sides of the equations in (5.10), according to the signs of $\mathrm{i}k_0$ or $\mathrm{i}k_B$. Thus for k_0 we have $\mathrm{i}k_0 = 2, -2c_+$ if $\mathrm{i}k_0 > 0$ and $\mathrm{i}k_0 = -2, 2c_+$ if $\mathrm{i}k_0 < 0$, while for k_B we have $\mathrm{i}k_B = 2\cos\rho, 2$ if $\mathrm{i}k_B > 0$ and $\mathrm{i}k_B = -2\cos\rho, -2$ if $\mathrm{i}k_B < 0$. However, the signs should be disregarded because for either sign each solution for a specific value of $|\mathrm{i}k_0|$ or $|\mathrm{i}k_B|$ represents a single 'bound-state' function η. Taking this into account, these special solutions contribute to Δ an additional piece:

$$T(\sin\rho - \sin\pi a_+) = \pi - \frac{\pi \sin\pi a_+}{\sin\rho}. \tag{5.19}$$

Assembling the various components leads to

$$\Delta = \pi - \frac{2}{\sin\rho}\left(\cos\rho + \cos\pi a_+ + \rho\sin\rho + \pi(a_+ - \tfrac{1}{2})\sin\pi a_+\right). \tag{5.20}$$

Recalling (5.12), and using (5.20), the quantum action defined in (5.11) is given by an expression of the form

$$S_{\mathrm{qu}} = \frac{4}{B}\left(\frac{\cos\rho}{\sin\rho} + \rho - \frac{\pi}{2}\right) + \frac{8\pi}{\beta^2}\left(\pi a_+ - \frac{\pi}{2}\right) + \frac{\Gamma(a_+, a_-)}{\sin\rho} \tag{5.21}$$

where Γ is independent of ρ,

$$\Gamma(a_+, a_-) = 2\pi\left(-\frac{2}{\beta^2}(1-c_+)(1+c_-) + \frac{4}{\beta^2} + \frac{c_+}{\pi} + (a_+ - \tfrac{1}{2})\sin\pi a_+\right).$$

Once the quantum action is determined, the quantum energy is defined by

$$\mathcal{E}_{\mathrm{qu}} = -\frac{\partial S_{\mathrm{qu}}}{\partial T} = \frac{\sin^2\rho}{\pi\cos\rho}\frac{\partial S_{\mathrm{qu}}}{\partial\rho} = -\frac{4}{\pi B}\cos\rho - \frac{\Gamma(a_+, a_-)}{\pi} \tag{5.22}$$

and the WKB quantization condition states that

$$W_{\mathrm{qu}} = S_{\mathrm{qu}} + T\mathcal{E}_{\mathrm{qu}} = \frac{4}{B}\left(\rho - \frac{\pi}{2}\right) + \frac{8\pi}{\beta^2}\left(\pi a_+ - \frac{\pi}{2}\right) = 2n\pi. \tag{5.23}$$

Where, n is a positive integer or zero†. Hence, the energies of the quantized boundary breather states are determined by a set of special angles ρ_n,

$$\rho_n = \frac{\pi}{2}\left(1 + nB - \frac{2\pi B}{\beta^2}(2a_+ - 1)\right) \tag{5.24}$$

and given by

$$\mathcal{E}_n = -\frac{4}{\pi B}\cos\rho_n - \frac{\Gamma}{\pi} = -\frac{4}{\pi B}\cos\frac{\pi}{2}\left(nB + 1 - \frac{2\pi B}{\beta^2}(2a_+ - 1)\right) - \frac{\Gamma}{\pi}. \tag{5.25}$$

† Including the imaginary solutions of (5.10) obviates the need for making the change $n \to n + \frac{1}{2}$, since we shall see below that the zero-point energy is automatically correct. In this sense, including the imaginary solutions leads more naturally to the expected result.

Note that as $\beta \to 0$, $\rho_n \to \pi(1 - a_+)$ independently of n. Thus, the frequencies collapse to the lowest allowed frequency, namely $\omega_0 = 2\sin\pi a_+$. On the other hand, in the same limit the energies are independent of β and non-zero,

$$\mathcal{E}_n \to \left(n + \tfrac{1}{2}\right)\omega_0. \tag{5.26}$$

This is precisely the spectrum of a harmonic oscillator vibrating at the fundamental frequency ω_0.

Using (5.25) the corresponding differences in the energy levels are given by

$$\mathcal{E}_{n+1} = \mathcal{E}_n + \frac{8}{\pi B}\sin\frac{\pi B}{4}\cos\frac{\pi}{2}\left(\frac{2\pi B}{\beta^2}(2a_+ - 1) - \left(n + \frac{1}{2}\right)B\right). \tag{5.27}$$

Comparing (5.27) with the outcome of the bootstrap calculation (4.8) allows us to identify the parameter E which appeared in the expression for the reflection factor (2.5). Thus, from the first excited level we deduce,

$$\begin{aligned} E(a_0, a_1, \beta) &= \frac{2\pi B}{\beta^2}(2a_+ - 1) - \frac{1}{2}B + 1 \\ &\equiv 2a_+\left(1 - \frac{B}{2}\right) = (a_0 + a_1)\left(1 - \frac{B}{2}\right) \end{aligned} \tag{5.28}$$

and then all the other levels match up without further restriction. This is in agreement with the suggestions made in [3].

As was pointed out previously in [6], the comparison with (4.8) also permits us to deduce an expression for $m(\beta)$, the mass of the sinh–Gordon particle:

$$m(\beta) = \frac{8}{\pi B}\sin\frac{\pi B}{4}. \tag{5.29}$$

6. Discussion

From the point of view of the classical reflection factor (2.7), it is clear that the parametrization (2.8) is the most appropriate. Moreover, the expressions for ε_0 and ε_1 in terms of a_0 and a_1 are invariant under independent changes of signs of either a_0 or a_1 and this symmetry is incorporated in the factors appearing in the denominator of (2.7). We expect that the symmetry under reversing the signs of a_0 and a_1 should persist in the quantum reflection factor, and indeed that was part of the thinking behind the conjectured forms for E and F presented in (2.10). In other words, having calculated E in (5.28) we can immediately deduce the partner expression for F.

We found that the general breathers exist as non-singular real solutions provided the parameters satisfy the constraints (3.11). It is gratifying to discover at the end of the calculation that the renormalization factor $1 - B/2$ in (2.10) for the real coupling β is restricted to lie in the range

$$0 \leqslant 1 - \tfrac{1}{2}B \leqslant 1$$

and thus has the effect of scaling the parameters but preserving the constraints (3.11). Note that with E and F given by (2.10) it is impossible to have E and F lying simultaneously in regions (4.3) for which the bound state poles of the reflection factor lie on the physical strip. Thus, the poles depending on E or F are alternatives and actually lead to the same tower of states simply seen differently in terms of the parameters a_0 and a_1.

Changing the sign of either a_0 or a_1 takes us outside the principal ranges of these parameters and obliges us to reformulate the breather solutions. In fact, there are no others besides those we have found, although the expressions for them will be a little different if we adopt a different principal region.

It is worth emphasizing that the expressions (2.10) incorporate a weak–strong-coupling duality [3] which extends that enjoyed by the S-matrix itself. If a new triple of coupling constants (a_0^*, a_1^*, β^*) is defined by

$$(a_0^*, a_1^*, \beta^*) = \frac{4\pi}{\beta^2}(a_0, a_1, \beta)$$

then

$$\begin{aligned} E(a_0^*, a_1^*, \beta^*) &= \frac{4\pi}{\beta^2}(a_0 + a_1)\frac{B}{2} \\ &\equiv (a_0 + a_1)(1 - B/2) = E(a_0, a_1, \beta) \end{aligned} \tag{6.1}$$

and similarly for F.

We must acknowledge that the semiclassical approximation, although non-perturbative, is not guaranteed to be exact. However, for the bulk sine–Gordon model, the DHN approach does give exact information concerning the spectrum and we would expect that the same should be true here. Unfortunately, alternative exact computations of the spectrum are not yet available for comparisons to be made.

As far as other models are concerned, it will be interesting to try out these methods in other cases where the boundary parameters form a continuous set. The simplest such example, in which there are two distinct one-parameter families of boundary parameters, is the model based on the $a_2^{(2)}$ root data [1].

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Appendix

In this appendix we shall give some information concerning the kinetic energy integral (3.16). The first step in evaluating the integral is to perform the integration over time. This is relatively easy and leads to

$$\begin{aligned} I(x) = \int_0^T \mathrm{d}t\, \dot{\phi}^2 = \frac{8\pi \sin\rho}{\beta^2}\bigg[&\frac{2C\sqrt{B^2 - A^2}}{AD - BC} + \frac{B}{\sqrt{B^2 - A^2}} \\ &- \frac{2A\sqrt{D^2 - C^2}}{AD - BC} + \frac{D}{\sqrt{D^2 - C^2}} \bigg] \end{aligned} \tag{A.1}$$

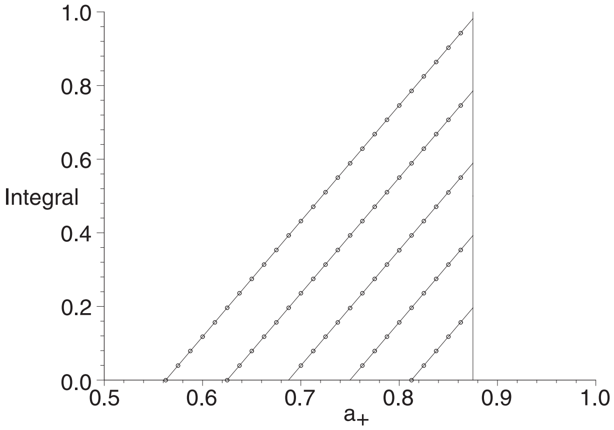

Figure A1. The integral as a function of a_+.

where the right-hand side is built from the components A, B, C, D which are defined by

$$\begin{aligned} A &= \frac{su}{q}(1-q^2)(1-rv) \\ B &= \frac{1}{q^2}\left((rv - s^2q^2u^2) + q^2(1-s^2q^2rvu^2)\right) \\ C &= -\frac{su}{q}(1-q^2)(1+rv) \\ D &= \frac{1}{q^2}\left(-(rv + s^2q^2u^2) + q^2(1+s^2q^2rvu^2)\right) \end{aligned} \tag{A.2}$$

with

$$u = e^{2x\cos\rho} \qquad v = e^{2x}. \tag{A.3}$$

The parameters q, r, s were defined previously in (3.7). Note, when $a_0 = a_1$ we have $r = 0$, allowing all of these expressions to simplify dramatically. Consequently, the integral of the right-hand side of (A.1) can be done by making a suitable change of variables. Thus, for $a_0 = a_1 = a$ we find:

$$\int_{-\infty}^{0} dx\, I(x) = \frac{8\pi}{\beta^2}(\rho - \pi(1-a)). \tag{A.4}$$

However, in the general case we have not found a change of variables which simplifies (A.1). Therefore, we have had to proceed numerically.

One check we have made is to differentiate (A.1) with respect to a_- and demonstrate that

$$\frac{\partial}{\partial a_-}\int_{-\infty}^{0} dx\, I(x) = 0 \tag{A.5}$$

provided $a_\pm$ lie within the ranges (3.11). Another check is to calculate the integral directly as a function of a_+ and ρ and demonstrate that it depends linearly on each of these parameters separately. The following plots generated by Maple indicate unambiguously the linear dependence on a_+ (figure A1) or on ρ (figure A2)

In figure A1 we have overlaid several plots. The dots represent the numerical values of the integral as a_+ varies from $1-\rho/\pi$ to $1-a_-$, in accordance with (3.11), for each of five different

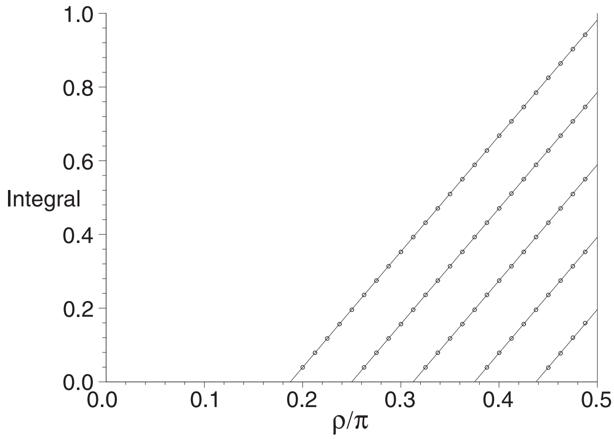

Figure A2. The integral as a function of ρ.

values of ρ (namely $\rho/\pi = (2+k)/16,\ k = 1, \ldots, 5$); the full line is simply the conjectured value of the integral given in (3.16) plotted as a function of a_+ for the same five values of ρ. In all plots, the factor of $8\pi/\beta^2$ has been ignored. The vertical line indicates the upper bound on a_+ for $a_- = \frac{1}{8}$. Figure A2 presents similar plots for the integral as a function of ρ/π in the range $1 - a_+$ to $\frac{1}{2}$ for five different values of a_+ (namely $a_+ = (8+k)/16,\ k = 1, \ldots, 5$). We regard the sets of parallel lines depicted in figures 1 and 2 as very convincing numerical evidence for (3.16).

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